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A note on the integrability of non-Hermitian extensions of Calogero-Moser-Sutherland models

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ABSTRACT: We consider non-Hermitian but PT -symmetric extensions of Calogero models, which have been proposed by Basu-Mallick and Kundu for two types of Lie algebras. We address the question of whether these extensions are meaningful for all remaining Lie algebras (Coxeter groups) and if in addition one may extend the models beyond the rational case to trigonometric, hyperbolic and elliptic models. We find that all these new models remain integrable, albeit for the non-rational potentials one requires additional terms in the extension in order to compensate for the breaking of integrability.

1. Introduction

Traditionally one considers quantum mechanical models and quantum field theories associated with Hermitian Hamiltonians, as in general they are guaranteed to have meaningful energy spectra, lead to conservation of probability densities under time evolution etc. Despite this apparent need for the Hamiltonian to be Hermitian, non-Hermitian Hamiltonian systems have been investigated for some time and found to be physical, as for instance in the context of level crossing [1, 2] and 1+1 dimensional quantum field theory [3, 4]. Fairly recent, the observation that the simple one-particle Hamiltonian with potential term $V = x^2(ix)^\nu$ for $\nu \geq 0$ possesses a real and positive spectrum [5] has triggered a sequence of investigations [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31]. One of the outcomes of these studies is the conjecture that Hermiticity is only a sufficient but not a necessary condition for the spectrum to be real and positive. Instead, one may simply demand the Hamiltonian to be PT -invariant in order to ensure the spectrum to be physical. Inevitably, non-Hermitian Hamiltonians will give rise to various other kinds of problems, such as an indefinite metric [20, 32], so that one is forced to give a proper meaning to unitary evolution etc. This particular problem is overcome by utilizing a new type of symmetry, which seems to be always present when the Hamiltonian is PT -invariant, and use it to define a new inner-product structure which yields positive

definite norms of the associated quantum states [7, 33]. Thus these type of non-Hermitian theories can be made consistent and are not in conflict with concepts of standard quantum mechanics, but can be regarded as meaningful extensions of them. Encouraged by these results similar investigations have also been extended to the realm of quantum field theories [34, 35, 36, 37, 38].

The sole requirement of PT -invariance allows to include also various types of momentum dependent terms into the potential of the Hamiltonian, which previously when demanding Hermiticity would have been excluded. Such type of models are attractive as they lead to interesting applications in condensed matter physics, because they are usually of anyonic nature and exhibit generalized exclusion statistics of Haldane type. For instance, one has considered extensions of simple harmonic oscillators of the form $V \sim ixp$ [39, 40], the non-linear Schrödinger equation perturbed by higher spatial dispersions [41], double delta potentials [42], etc. Motivated by this, Basu-Mallick and Kundu [43] have extended the above mentioned investigations from one to many-particle systems and proposed a new type of model which constitutes a non-Hermitian extension of the rational A_ℓ -Calogero models [44]

$$\mathcal{H}_{BK} = \frac{p^2}{2} + \frac{\omega^2}{2} \sum_i q_i^2 + \frac{g^2}{2} \sum_{i \neq k} \frac{1}{(q_i - q_k)^2} + i\tilde{g} \sum_{i \neq k} \frac{1}{(q_i - q_k)} p_i \quad g, \tilde{g} \in \mathbb{R}, q, p \in \mathbb{R}^{\ell+1}, \quad (1.1)$$

where $p_i \equiv -i\partial/\partial q_i$. Clearly this Hamiltonian is no longer Hermitian, but its extension remains unchanged when transformed under a time-reversal and a subsequent parity transformation

$$P: p_j \mapsto -p_j, \quad q_j \mapsto -q_j \quad T: p_j \mapsto -p_j, \quad q_j \mapsto q_j, \quad i \mapsto -i, \quad (1.2)$$

i.e. the Hamiltonian \mathcal{H}_{BK} is PT -invariant. Subsequently, various aspects of the model have been studied [45, 46, 47] and intriguingly it was found that in these models the exclusion and exchange parameter differ, unlike in the conventional Calogero models, that is the case $\tilde{g} = 0$, where they are identical.

With regard to the standard Calogero models, there are four conceivable generalizations for the Hamiltonian \mathcal{H}_{BK} . First a fairly trivial one, a formulation independent of the explicit representation for the roots of the A_ℓ -Weyl group, second a generalization of the possible potentials including those which are trigonometric, hyperbolic and elliptic, third a generalization to Lie algebras (or better Coxeter groups) other than A_ℓ and fourth the possibility to include more coupling constants. A generalization of (1.1) to Calogero models of B_ℓ -type has been studied already in [45]. An important question to answer is whether these extended models remain integrable in a similar way as their original counterparts or whether the additional term destroys this valuable property. Despite the fact that the issue of Hermiticity is mainly relevant in the quantum theory, we investigate here the classical integrability of these models. Most likely this will also be important in the quantum theory, as it is well known that in these type of models the quantum theories inherit often many properties of their classical counterparts, especially the feature of being integrable or not. The main purpose of this note is to establish which type of extensions of the Calogero-Moser-Sutherland (CMS) models [44, 48, 49, 50] preserve integrability.

2. Integrability of non-Hermitian PT-invariant extensions of CMS-models

For simplicity we ignore for the time being the confining term in (1.1), that means we set $\omega = 0$, and investigate the following generalization of the Basu-Mallick Kundu model with regard to all four of the above mentioned possible generalizations

$$\mathcal{H} = \mathcal{H}_{\text{Cal}} + \mathcal{H}_{PT} = \frac{p^2}{2} + \frac{1}{2} \sum_{\alpha \in \Delta} g_\alpha^2 V(\alpha \cdot q) + \frac{i}{2} \sum_{\alpha \in \Delta} \tilde{g}_\alpha f(\alpha \cdot q)(\alpha \cdot p). \quad (2.1)$$

Here Δ is any root system invariant under Coxeter transformations. We further assume that the potential and the function $f(x)$ in \mathcal{H}_{PT} are related as $V(x) = f^2(x)$. Besides the rational case $f(x) = 1/x$ considered previously for the A_ℓ and B_ℓ -case [43, 45], we also want to consider the remaining possibilities of the CMS-models, the trigonometric case $f(x) = 1/\sin x$, the hyperbolic case $f(x) = 1/\sinh x$ and in particular the elliptic case $f(x) = 1/\text{sn } x$. The Hamiltonian \mathcal{H}_{Cal} in (2.1) is the usual representation independent, meaning the roots, formulation of the CMS models. The equality of the last term in (1.1) and the last term in (2.1) for $\tilde{g} = g_\alpha$ is directly seen when the simple roots of A_ℓ are expressed in their standard $(\ell + 1)$ -dimensional representation, see e.g. [51], $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $1 \leq i \leq \ell$, with $\varepsilon_i \cdot \varepsilon_j = \delta_{ij}$. Having a formulation independent of the representation of the roots, we can next address the question of how many different coupling constants are permitted. A standard argument is to demand the invariance of the potential under the action of the Coxeter group. As the Coxeter transformations preserve the inner product structure, roots of the same length are mapped into each other, such that the roots can be divided into the two subsets of long and short roots, $\Delta = \Delta_s \cup \Delta_l$, which are left invariant by the Coxeter transformations. This means based on demanding invariance, the extended models possess also two independent coupling constants like their CMS counterparts

$$g_\alpha = \begin{cases} g_s & \text{for } \alpha \in \Delta_s \\ g_l & \text{for } \alpha \in \Delta_l \end{cases} \quad \text{and} \quad \tilde{g}_\alpha = \begin{cases} \tilde{g}_s & \text{for } \alpha \in \Delta_s \\ \tilde{g}_l & \text{for } \alpha \in \Delta_l \end{cases}. \quad (2.2)$$

Demanding integrability often restricts this choice further, e.g. [50, 52, 53].

For the rational version of the Calogero models, i.e. when $f(x) = 1/x$, we note next the crucial property

$$\eta^2 = \alpha_s^2 \tilde{g}_s^2 \sum_{\alpha \in \Delta_s} V(\alpha \cdot q) + \alpha_l^2 \tilde{g}_l^2 \sum_{\alpha \in \Delta_l} V(\alpha \cdot q) \quad \text{with} \quad \eta = \frac{1}{2} \sum_{\alpha \in \Delta} \tilde{g}_\alpha f(\alpha \cdot q) \alpha. \quad (2.3)$$

Before using (2.3), let us first consider an argument to establish that it actually holds. The identity implies that when computing η^2 all terms involving products of the form $f(\alpha \cdot q)f(\beta \cdot q)(\alpha \cdot \beta)$ for which $\alpha \neq \beta$ cancel each other. To see this we gather all terms in triplets involving two arbitrary roots α, β and a third root which is their sum $\gamma = \alpha + \beta$. It may happen though that not all three terms of this type appear in the product $\eta \cdot \eta$ due to the fact that either $\alpha \cdot \beta = 0$ or $\beta \cdot \gamma = 0$. In that case we can suitably add several of the missing terms in the hope that overall the additional terms sum up to zero. It turns out that one may always group the terms conveniently and cancel them by means of four

basic identities. Keeping our discussion representation independent, these identities can be characterized by the value of the inner product of the co-roots $\hat{\alpha} = 2\alpha/\alpha^2$ and $\hat{\beta} = 2\beta/\beta^2$. For $\alpha, \beta, \gamma = \alpha + \beta$ we find the relations

$$-\frac{\alpha \cdot \alpha}{(\alpha \cdot q)(\beta \cdot q)} + \frac{\alpha \cdot \gamma}{(\alpha \cdot q)(\gamma \cdot q)} + \frac{\beta \cdot \gamma}{(\beta \cdot q)(\gamma \cdot q)} = 0 \quad \text{for } \hat{\alpha} \cdot \hat{\beta} = 0, \quad (2.4)$$

$$\frac{\alpha \cdot \beta}{(\alpha \cdot q)(\beta \cdot q)} + \frac{\alpha \cdot \gamma}{(\alpha \cdot q)(\gamma \cdot q)} + \frac{\beta \cdot \gamma}{(\beta \cdot q)(\gamma \cdot q)} = 0 \quad \text{for } \hat{\alpha} \cdot \hat{\beta} = 1, \quad (2.5)$$

$$\frac{\alpha \cdot \beta}{(\alpha \cdot q)(\beta \cdot q)} + \frac{\alpha \cdot \gamma}{(\alpha \cdot q)(\gamma \cdot q)} + \frac{\beta \cdot \beta}{(\beta \cdot q)(\gamma \cdot q)} = 0 \quad \text{for } \hat{\alpha} \cdot \hat{\beta} = 2, \quad (2.6)$$

$$\frac{\alpha \cdot \beta}{(\alpha \cdot q)(\beta \cdot q)} + \frac{\alpha \cdot \gamma}{(\alpha \cdot q)(\gamma \cdot q)} - \frac{3\beta \cdot \gamma}{(\beta \cdot q)(\gamma \cdot q)} = 0 \quad \text{for } \hat{\alpha} \cdot \hat{\beta} = 3, \quad (2.7)$$

which may be used successively to establish (2.3). Relation (2.5), which applies whenever we have three roots of the same length is most obvious to use as it involves the terms appearing in the sum when computing the product $\eta \cdot \eta$ and it is just a matter of grouping the term together. This involves a non-trivial counting as it requires the precise knowledge of which inner product of the two roots are non-vanishing and also the information that after the re-grouping there are no leftovers. Unfortunately we are not aware of a case independent proof for this. However, we systematically verified this for many Coxeter groups, and based on that we assume that (2.3) holds in general. To sustain this we present here just some selected examples:

As a representative for root systems involving only roots of one length, such as all those related to simply laced Lie algebras, we consider the A_3 -case. The six positive roots in this case are

$$\Delta_{A_3}^+ = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4 = \alpha_1 + \alpha_2, \alpha_5 = \alpha_2 + \alpha_3, \alpha_6 = \alpha_1 + \alpha_2 + \alpha_3\}. \quad (2.8)$$

We abbreviate now $\hat{f}_i := \alpha_i/(\alpha_i \cdot q)$, $\alpha^2 = \alpha_i^2$ for $1 \leq i \leq 6$ and $g^2 = g_s^2 = g_l^2$. Then using that the only non-vanishing off-diagonal entries in the Cartan matrix $K_{ij} = 2\alpha_i \cdot \alpha_j/\alpha_j^2$ are $K_{12} = K_{21} = K_{23} = K_{32} = -1$, we compute

$$\begin{aligned} \eta^2 &= g^2 \sum_{k=1}^6 \frac{\alpha^2}{(\alpha_k \cdot q)^2} + \alpha^2 g^2 \left(\hat{f}_1 \cdot \hat{f}_2 + \hat{f}_1 \cdot \hat{f}_4 + \hat{f}_2 \cdot \hat{f}_4 + \hat{f}_2 \cdot \hat{f}_3 + \hat{f}_2 \cdot \hat{f}_5 + \hat{f}_3 \cdot \hat{f}_5 \right. \\ &\quad \left. + \hat{f}_1 \cdot \hat{f}_5 + \hat{f}_5 \cdot \hat{f}_6 + \hat{f}_1 \cdot \hat{f}_6 + \hat{f}_4 \cdot \hat{f}_3 + \hat{f}_4 \cdot \hat{f}_6 + \hat{f}_3 \cdot \hat{f}_6 \right) \end{aligned} \quad (2.9)$$

$$= g^2 \sum_{k=1}^6 \frac{\alpha^2}{(\alpha_k \cdot q)^2}. \quad (2.10)$$

We organized the last terms already successively into triplets in such a way that it is easy to see that they all cancel directly by means of (2.5).

The non-simply laced cases are less straightforward. The simplest example involving long and short roots with $\hat{\alpha} \cdot \hat{\beta} = 2$ is the B_2 -case. The four positive roots for this are

$$\Delta_{B_2}^+ = \Delta_l^+ = \{\alpha_1, \alpha_3 = \alpha_1 + 2\alpha_2\} \cup \Delta_s^+ = \{\alpha_2, \alpha_4 = \alpha_1 + \alpha_2\}. \quad (2.11)$$

The B_2 -Cartan matrix has entries $K_{12} = -2$ and $K_{21} = -1$, from which we compute

$$\eta^2 = \sum_{k=1}^4 \frac{g_k^2 \alpha_k^2}{(\alpha_k \cdot q)^2} + \alpha_l^2 g_l g_s \left(\hat{f}_1 \cdot \hat{f}_2 + \hat{f}_1 \cdot \hat{f}_4 + \hat{f}_2 \cdot \hat{f}_3 + \hat{f}_3 \cdot \hat{f}_4 \right) \quad (2.12)$$

$$= \sum_{k=1}^4 \frac{g_k^2 \alpha_k^2}{(\alpha_k \cdot q)^2} + \alpha_l^2 g_l g_s \left(-\frac{\alpha_2^2}{(\alpha_2 \cdot q)(\alpha_4 \cdot q)} + \frac{\alpha_2^2}{(\alpha_2 \cdot q)(\alpha_4 \cdot q)} \right) \quad (2.13)$$

$$= \sum_{k=1}^4 \frac{g_k^2 \alpha_k^2}{(\alpha_k \cdot q)^2}. \quad (2.14)$$

Here we used (2.6) to obtain the penultimate term in (2.13) from the third and fourth last term in (2.12) and identity (2.4) to obtain the last term in (2.13) from the last two terms in (2.12). Overall the required terms to complete the identities (2.4), (2.6) add up to zero.

Identity (2.6) is required for the G_2 -case. The six positive roots are now

$$\begin{aligned} \Delta_{G_2}^+ &= \Delta_s^+ \cup \Delta_l^+ \\ &= \{\alpha_1, \alpha_3 = \alpha_1 + \alpha_2, \alpha_4 = 2\alpha_1 + \alpha_2\} \cup \{\alpha_2, \alpha_5 = 3\alpha_1 + \alpha_2, \alpha_6 = 3\alpha_1 + 2\alpha_2\}. \end{aligned} \quad (2.15)$$

The G_2 -Cartan matrix has entries $K_{12} = -1$ and $K_{21} = -3$, which yields

$$\begin{aligned} \eta^2 &= \sum_{k=1}^6 \frac{g_k^2 \alpha_k^2}{(\alpha_k \cdot q)^2} + \alpha_s^2 g_s^2 \left(\hat{f}_1 \cdot \hat{f}_3 + \hat{f}_1 \cdot \hat{f}_4 + \hat{f}_3 \cdot \hat{f}_4 \right) + \alpha_l^2 g_l^2 \left(\hat{f}_2 \cdot \hat{f}_5 + \hat{f}_2 \cdot \hat{f}_6 + \hat{f}_5 \cdot \hat{f}_6 \right) \\ &\quad + \alpha_l^2 g_s g_l \left(\hat{f}_1 \cdot \hat{f}_2 + \hat{f}_2 \cdot \hat{f}_3 + \hat{f}_1 \cdot \hat{f}_5 + \hat{f}_4 \cdot \hat{f}_5 + \hat{f}_3 \cdot \hat{f}_6 + \hat{f}_4 \cdot \hat{f}_6 \right) \end{aligned} \quad (2.16)$$

$$= \sum_{k=1}^6 \frac{g_k^2 \alpha_k^2}{(\alpha_k \cdot q)^2} - 3\alpha_l^2 g_s g_l \left(\hat{f}_1 \cdot \hat{f}_3 + \hat{f}_1 \cdot \hat{f}_4 + \hat{f}_3 \cdot \hat{f}_4 \right) \quad (2.17)$$

$$= \sum_{k=1}^6 \frac{g_k^2 \alpha_k^2}{(\alpha_k \cdot q)^2}. \quad (2.18)$$

Here we employed (2.5) to cancel the last two triplets in (2.16). In the step from (2.16) to (2.17) we used (2.7) and then cancel the last three terms in (2.16) by means of (2.5).

In a similar manner as for the presented examples we may establish (2.3) for the remaining cases. Unfortunately, we are not aware of a case independent argument to prove this in complete generality. Furthermore, we note that (2.3) does not hold in general for the non-rational potentials.

We return now to our main line of argument and employ the identity (2.3) to re-write the Hamiltonian (2.1) as a conventional Calogero model with shifted momenta

$$\mathcal{H} = \frac{1}{2}(p + i\eta)^2 + \frac{1}{2} \sum_{\alpha \in \Delta} \hat{g}_\alpha^2 V(\alpha \cdot q). \quad (2.19)$$

together with some re-defined coupling constants

$$\hat{g}_\alpha^2 = \begin{cases} g_s^2 + \alpha_s^2 \tilde{g}_s^2 & \text{for } \alpha \in \Delta_s \\ g_l^2 + \alpha_l^2 \tilde{g}_l^2 & \text{for } \alpha \in \Delta_l \end{cases}. \quad (2.20)$$

Next we recall [54] that classical integrability may be established by formulating Lax pair operators L and M as functions of the dynamical variables q_i and p_i , which satisfy the Lax equation $\dot{L} = [L, M]$, upon the validity of the classical equation of motion resulting from the corresponding Hamiltonian. Taking the observation on board that the extended model and the ordinary Calogero model only differ by a specific shift in the momenta and a re-definition of the coupling constants, it is straightforward to see that this also holds for the Lax operators. Thus we take the conventional Lax operators for the CMS models and simply replace $p \rightarrow p + i\eta$. One may then check directly that

$$L = (p + i\eta) \cdot H + i \sum_{\alpha \in \Delta} \hat{g}_\alpha f(\alpha \cdot q) E_\alpha \quad \text{and} \quad M = m \cdot H + i \sum_{\alpha \in \Delta} \hat{g}_\alpha f'(\alpha \cdot q) E_\alpha \quad (2.21)$$

fulfills the Lax equation with the constraint

$$\dot{q}_j = \frac{\partial \mathcal{H}}{\partial p_j} = p_j + i\eta_j \quad \text{and} \quad \dot{p}_j = -\frac{\partial \mathcal{H}}{\partial q_j} = -\frac{\partial \mathcal{H}_{\text{Cal}}}{\partial q_j} - i\eta_j, \quad (2.22)$$

if $\dot{L}_{\text{Cal}} = [L_{\text{Cal}}, M_{\text{Cal}}]$. We choose here as convention the Cartan-Weyl basis commutation relations

$$[H_i, H_j] = 0, \quad [H_i, E_\alpha] = \alpha^i E_\alpha, \quad [E_\alpha, E_{-\alpha}] = \alpha \cdot H, \quad [E_\alpha, E_\beta] = \varepsilon_{\alpha, \beta} E_{\alpha+\beta}. \quad (2.23)$$

which is compatible with $\text{tr}(H_i H_j) = \delta_{ij}$, $\text{tr}(E_\alpha E_{-\alpha}) = 1$. The vector m can be specified as a function of the structure constants $\varepsilon_{\alpha, \beta}$ and the potential. As all additional terms resulting from the shift $i\eta$ cancel in the Lax equation, the requirements imposed by integrability, i.e. the validity of the Lax equation are exactly the same as in the non-extended models. Note that the Lax equation is solved directly only for the A_ℓ -algebra, but for other algebras we have to follow the reduction procedures as indicated for instance in [55, 50, 52, 53]. Having now established that L and M in (2.21) are meaningful Lax operators for the extended Hamiltonian (2.19), we may compute backwards and expand the kinetic term such that we simply obtain

$$\mathcal{H} = \frac{1}{2} p^2 + \frac{1}{2} \sum_{\alpha \in \Delta} \hat{g}_\alpha^2 V(\alpha \cdot q) + i\eta \cdot p - \frac{1}{2} \eta^2. \quad (2.24)$$

Note that we did not make any assumption on the potential, such that (2.24) is a non-Hermitian *integrable* extension for CMS models for all Coxeter groups, including besides the rational also trigonometric, hyperbolic and elliptic potentials. We observe that when one wishes to preserve integrability for the non-rational potentials one can not simply extend the models by the term $i\eta \cdot p$, but one also has to add the momentum independent term $-\eta^2/2$ in order to compensate for the integrability breaking effect of that term.

3. Conclusions

We have demonstrated that the non-Hermitian extensions for the rational Calogero model proposed by Basu-Mallick and Kundu for the A_ℓ and B_ℓ Coxeter groups can be generalized to all remaining groups in such a way that they are classically integrable. The identity (2.3)

is crucial in this context and it would be interesting to have a rigorous generic, i.e. case independent proof for it. The identity for η^2 ensures that the extended Hamiltonian differs from the original Calogero model only by the one term $i\eta \cdot p$. This simplicity can only be maintained when the potential is rational. However, adding one more term as proposed in (2.24) one obtains integrable extensions for all CMS-models. One should stress that the above argument does not exclude yet the possibility that (2.24) might be integrable even for non-rational potentials when the last term is dropped. Nonetheless, it establishes that when we include this term they are integrable for sure. It would be very interesting to carry out further studies on these new models along the lines previously followed in [45, 46, 47] and beyond.

Furthermore, it would be interesting to extend the analysis in [56], where the question of solvability (\neq integrability) of some of the discussed models has been addressed.

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